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n-Person Dynamic Strategic Market Games

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Abstract We present a discrete n-person model of a dynamic strategic market game. We show that for some values of the discount factor the game possesses a stationary equilibrium where all the players make high bids. Within the class of all the high-bidding strategies we distinguish between two classes of more and less aggressive ones. We show that the set of discount factors for which these more aggressive strategies form equilibria shrinks as n goes to infinity. On the other hand, the analogous set for the less aggressive strategies grows to the whole interval $(0, 1)$ as n grows to infinity. Further we analyze the properties of the value function corresponding to these high-bidding equilibria. We also give some numerical examples contradicting some other properties that seem intuitive.

Keywords Stochastic game · Strategic market game · n-person game · Stationary equilibrium · Finite strategy space

1 Introduction

We consider n players—each holding an integral amount of money, competing for portions of some nondurable commodity. At each of infinitely many stages of the game, one unit of the good is brought to the market, and players bid integral parts of their money for it. If a bid of a player is accepted, he pays for the good and receives a share of the good according to some fixed rule of distribution. The sum of all the payments is returned to the players at the end of the stage, each player receiving a random share according to some probability distribution. The utility of the share of the good consumed by a player is measured by some concave utility function. The

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utilities in subsequent periods of the game are discounted using some fixed discount factor β.

The main problem for a player in this setting is to decide whether to spend more now or to save the money for the future consumption. The decision has to take into account both the magnitude of β (with smaller β it is intuitively more justified to save less for the future) and the fact that the situation in the game may change in a direction in which the same amount of good may be bought either at a lower or at a higher price—which would make saving either more or less sensible. The fact that the model is stochastic adds uncertainty as another important factor in the decision making.

The models of this type were extensively studied in the literature, beginning with the model of Shubik and Whitt [11], where a nonstochastic proportional-reward game was considered. They showed that in their setting, the strategy of spending all the money for the good becomes optimal for all the players as n becomes large enough. Most of the further literature on the same topic breaks into two groups. In the first one ([9, 10, 12, 13]) only 2-player games are considered, and in most of the papers [9, 10, 13] only results about the form of equilibria for β small or large enough are proved. Only [12] provides some information about the structure of the equilibria for all the values of the discount factor. We should note here that in [10] the authors claim that some of the results generalize to n-person games, however we are not given information which ones and to what extent.

The second group of models were the ones with continuum of players [2, 4, 5]. Here the authors both provided general results characterizing stationary competitive equilibria in their games and gave some simple examples illustrating them. [4] is a relatively simple model, which can be viewed as a counterpart of the models studied for 2 players, while [2, 5] are its extensions—[5] with borrowing and lending allowed and [2] with a possible bankruptcy.

The only results about stochastic models with a number of players finite, but bigger than 2 appear in [6]. In Chap. 3 there the author treats the n-person counterpart of the proportional reward game from [10], providing results about the form of equilibria for small values of β and discussing the situation when β approaches 1. In this paper we provide a detailed analysis of n-person strategic market games with a "winner-takes-all" payoff, analyzed before for a 2-player case in [10, 12] (in which an auction is performed at every stage of the game). Unfortunately we have not proved general results about the structure of equilibria for any n and any β. Nevertheless, similarly as in [11], the results we prove are strong enough to imply an asymptotic result, true for any value of β and n large enough. In our analysis we concentrate on the variant of a "winner-takes-all" model studied for 2 players in [12], with a fixed simple rule of redistribution of the money, but we provide some insights of how these results extrapolate to more general settings. In our focus on this auction-based model we also have in mind that all the models studied for large number (continuum) of players were proportional-reward games (as this kind of auction model has no counterpart with continuum of players), and so this is the first analysis of this kind of model for a number of players bigger than 2.

The organization of the paper is as follows: In Sect. 2 we describe formally our model and introduce the notation used throughout the paper. In Sect. 3 we present all our main results and give a detailed analysis of their implications and possible extensions. The final section contains all the proofs.

2 Preliminaries

In this section we provide all the notation and definitions used in the subsequent part of the paper. In particular, we define special types of strategies that will be proved to form equilibria in some cases in the next section.

We begin with a general description of the game under investigation as a stochastic game.

1. State space of the game is the state of n-tuples of nonnegative integers summing up to M,

$$S=\left\{x=(x_1,\dots,x_n)\in\mathbb{N}^n:\sum_{i=1}^n x_i=M\right\}.$$

   k-th coordinate of the state variable x, x_k, is the amount of money player k possesses at a specified stage of the game.

   In the subsequent part of the paper we will also use the notation $x_{(k)}$, to denote the k-th largest coordinate of x. If two coordinates have the same value then also a number of $x_{(k)}$s have the same value, for example if $x=(5,5,3)$, $x_{(1)}=x_{(2)}=5$ and $x_{(3)}=3$.
2. The set of actions available to player i in state x will be denoted by $A_i(x)$. Of course

$$A_i(x)=\{0,1,\dots,x_i\}.$$

   The set of all the actions available in any state will be denoted by $A_i=\bigcup_{x\in X}A_i(x)$. For the actions we shall also use the notation $a_{(k)}$ to denote the k-th largest action.
3. Daily reward to player i in state x, when players use actions $(a_1,a_2,\dots,a_n)$ will be denoted by $r_i(x,a_1,a_2,\dots,a_n)$. It depends on the rule of distribution of the good after the bids are made, and therefore we shall define it after we introduce this rule.
4. The law of motion between states q also depends on the rule of distribution of the good, as well as on the second rule—rule of redistribution of money, and thus will also be defined later.

In the next couple of definitions we make precise, what kind of solutions we look for in our game. A *strategy* of player i is a sequence $\bar{\pi}_i=(\pi_i^1,\pi_i^2,\dots)$, where $\pi_i^t\in P(A_i)(x^t)$ is a distribution over the set of actions available to player i on the t-th stage of the game in state x^t, depending on partial history of the game up to its t-th stage $h_t=(x^1,a_1^1,\dots,a_n^1,\dots,x^{t-1},a_1^{t-1},\dots,a_n^{t-1},x^t)$. We say that a strategy $\bar{f}_i=(\pi_i^1,\pi_i^2,\dots)$ of player i is *stationary*, when for every moment t, $\pi_i^t=f_i(x^t)$ for some $f_i:S\to P(A)$. In the sequel we shall identify stationary strategy $\bar{f}_i$ with f_i. The set of all the stationary strategies of player i will be denoted by F_i.

The vector of strategies of the players $f=(f_1,\dots,f_n)$ together with the initial state x and the law of motion determine the unique probability distribution $P_{x,f}$ on the infinite histories of the game $(x,a_1^1,\dots,,a_n^1,x^2,a_1^2,\dots,a_n^2,\dots)$. The *expected discounted reward* to player i is defined by the formula:

$$J_i(x,f)=E_{x,f}\left(\sum_{t=1}^{\infty}\beta^{t-1}r_i\big(x^t,a_1^t,\dots,a_n^t\big)\right).$$

Throughout the paper, the discount factor β is assumed to satisfy $0<\beta<1$.

A profile of (stationary) strategies $f=(f_1,\dots,f_n)$ is called a *(stationary) equilibrium* for the discounted stochastic game, iff for every $x\in S$ and every player i,

$$J_i(x,f)\geq J_i\big(x,(f_{-i},\pi)\big),$$

for every strategy π of player i. (Here and in the sequel the notation (y_{-i},z) denotes vector $y=(y_1,\dots,y_n)$ with its i-th coordinate replaced by z. Similarly, (y_{-i-j},z_1,z_2) will denote y with its i-th coordinate replaced by z_1 and j-th coordinate replaced by z_2 and so on for a bigger number of coordinates).

Next we define three special types of stationary strategies that will appear in our results (here and in the sequel $\delta[z]$ denotes a degenerate probability distribution concentrated in point z).

A strategy f_i of player i is a *bold* strategy if

$$f_i(x)=\delta[x_i]\quad \text{if } x_i\leq x_{(2)}\quad\text{and}$$
$$\operatorname{supp}\big(f_i(x)\big)\subset\{x_{(2)}+1,\dots,x_i\}\quad \text{if } x_i>x_{(2)}.$$

A strategy f_i of player i is a *weakly bold* strategy if

$$f_i(x)=\delta[x_i]\quad \text{if } x_i\leq x_{(2)}\quad\text{and}$$
$$\operatorname{supp}\big(f_i(x)\big)\subset\{x_{(2)},\dots,x_i\}\quad \text{if } x_i>x_{(2)}.$$

The sets of bold and weakly bold strategies of player i will be denoted respectively by F_i^B and F_i^{WB}.

The strategy σ_i of player i defined by the formula:

$$\sigma_i(x)=\begin{cases}\delta[x_{(2)}+1] & \text{if } x_i=x_{(1)}\neq x_{(2)},\\ \delta[x_i] & \text{otherwise}\end{cases}$$

is called the *simple bold* strategy.

Now we present some more specific features of the model:

1. Rule of distribution of the good

   There are two main rules of distribution of the good considered in the literature: winner-takes-all rule [9, 10, 12, 13] and proportional reward rule [6, 9–11]. In our paper we consider only the first one. In *winner-takes-all game* at every stage of the game an auction is made. The winner of this auction (a player whose bid was the highest) receives whole of the good and has to pay for it. In case there are two or more players with the highest bid, one of these players is chosen with a chance move (with equal probability of being chosen for each player), and this player consumes the whole portion and pays for it. Then (after the consumption) all of the money paid is redistributed among the players according to some probability distribution.

Daily reward to player i is thus defined by the formula

$$r_i(x, a_1, \dots, a_n) = \begin{cases} \frac{1}{|\{j : a_j = a_i\}|} u_i(1) & \text{when } a_i = \max_j a_j, \\ u_i(0) & \text{when } a_i < \max_j a_j, \end{cases}$$

where u_i is a utility function of player i. We will assume every u_i is concave nondecreasing and satisfies $u_i(0) = 0$.

2. The law of motion between states

The law of motion between states for such a game is determined by the probability distribution, according to which the money spent for good is redistributed $p(a_{(1)})$, where $a_{(1)} = \max_j a_j$ is the amount of money spent for the good. $p(a_{(1)})$ is a distribution on the set

$$\left\{ (y_1, \dots, y_n) \in \mathbb{N}^n : \sum_{i=1}^{n} y_i = a_{(1)} \right\}.$$

Let x be current state of the game, $Y(a_{(1)})$—a random vector with distribution $p(a_{(1)})$, and $Z(a)$—a random 0-1 vector with value 1 for exactly one element of the set $\{j : a_j = a_{(1)}\}$ and equal probability for each of them. Then a random vector with distribution $q(x, a_1, \dots, a_n)$, $X' = (X'_1, \dots, X'_n)$ is given by

$$X'_k = \begin{cases} x_k - a_k(Z(a))_k + (Y(a_{(1)}))_k & \text{if } a_k = a_{(1)}, \\ x_k + (Y(a_{(1)}))_k & \text{otherwise.} \end{cases}$$

It is tricky to write q with a closed-form formula for a general distribution $p(a_{(1)})$. However, it is possible in a specific case. In the case considered in the next section, where the whole amount of money is redistributed to one player with equal probabilities for each of them,[1] transition probability is given for any $(x, a_1, \dots, a_n)$ by the following formula:

$$q(x, a_1, \dots, a_n) = \frac{1}{n}\delta[x] + \frac{1}{n}\sum_{k \neq i} \delta\big[(x_{-i-k}, x_i - a_i, x_k + a_i)\big], \quad \text{where } a_i = a_{(1)}.$$

Before we present our main results, we shall introduce two additional notions appearing in the subsequent theorems:

[1] Similar results hold, and can be proved analogously, for a more general case, which will be referred to in one of the subsections of the next section. The focus on this particular (and relatively simple) case is motivated by the fact that it allows simplifying some of the notation, as well as giving the results in more detail.

Definition 1 A strategy profile $f=(f_1,f_2,\dots,f_n)$ is *symmetric* if

(a) For any fixed i and $j,k\neq i$, $f_i(x_{-j-k},x_k,x_j)=f_i(x)$.
(b) For any $i\neq j$ and two states x,y, if $x=(y_{-i-j},y_j,y_i)$, then $f_i(x)=f_j(y)$.

Definition 2 A profile of strategies of players other than i, g_{-i} is *symmetric* to a strategy f_i of player i (we will denote it by[2] $g_{-i}=\mathrm{sym}(f_i)$) if $f:=(g_{-i},f_i)$ is a symmetric profile.

3 The Results

In the first main result of the paper we show when bold (weakly bold) selectors form an equilibrium in the game under consideration.

Theorem 1

(i) *For every $n\geq 3$ and every $\beta\leq\frac{1}{3}$, the game possesses a symmetric stationary equilibrium $f=(f_1,f_2,\dots,f_n)$, where f_i are bold strategies.*
(ii) *For every $n\geq 3$ and every $\beta\leq 1-\sqrt[3]{\frac{2n^2-6n+4}{n^3}}$, the game possesses a symmetric stationary equilibrium $f=(f_1,f_2,\dots,f_n)$, where f_i are weakly bold strategies.*
(iii) *For every $\beta\geq\frac{1}{2}$ there exists an n_β such that for every $n\geq n_\beta$, there is no strategy profile where the strategies of all the players are bold which is a stationary equilibrium in the game.*

Remark 1 One may think that in the result of part (i) of the theorem *some* bold strategy f_i could be replaced by the *simple bold selector* σ_i (which is a direct counterpart of bold selectors considered in [10, 12]). The numerical example below shows that this is not the case.

Consider the game with 3 symmetric players, $u_i(x)=x$, $\beta=0.3$ and $M=13$. The reward of a player using the simple bold strategy against simple bold strategies of all the other players in this game is in state $(7,2,4)$ equal to 1.12916. On the other hand, if he uses another bold strategy, prescribing him to bid 7 in state $(7,2,4)$, and use the simple bold strategy in any other state, his reward in $(7,2,4)$ is equal to 1.13032. This obviously implies that the profile of simple bold strategies is not an equilibrium in such a game.[3]

Remark 2 The third part of the theorem could be written in a more precise form if we considered some concrete bold strategy instead of *any*. For example, if we considered the simple bold selector, we could prove that for any $\beta>\frac{n}{2n-3}$ the profile $(\sigma,\dots,\sigma)$ cannot be an equilibrium in the game under consideration.

However, as seen from the example in Remark 1, there are situations when indeed only some other bold selectors form an equilibrium in the game.

[2]It is straightforward to show that there may only be one strategy of player j symmetric to some given strategy of i, thus the notation $g_{-i}=\mathrm{sym}(f_i)$ (instead of $g_{-i}\in\mathrm{sym}(f_i)$) is well justified.

[3]The same is true for smaller values of β.

A natural consequence of this theorem is the following:

Corollary 1 *For every $\beta < 1$ there exists an n such that n-person game possesses a symmetric stationary equilibrium $f = (f_1, f_2, \dots, f_n)$, where f_i are weakly bold strategies.*

In the second main result we present main properties of the reward functions in equilibrium.

Theorem 2 *Let $\beta \in (0, 1)$ and let V_i^* be the reward function of player i when all the players apply equilibrium strategies from Theorem 1. Then*:

(i) *For any state x and every i, $V_i^*(x) \le \frac{u_i(1)}{1-\beta}$.*
(ii) *If x is such that $x_i < x_{(2)}$ then $V_i^*(x) \le \frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)}$.*
(iii) *If x is such that $x_i < 2x_{(2)}$ then*

$$V_i^*(x) \le \frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}.$$

(iv) *Suppose that $\Delta \le x_{(2)}$ and that $x_i = x_{(1)}$ and $x_k = x_{(2)}$. Then for any $j \ne i, k$*:

$$\begin{aligned} &|V_i^*(x_{-i-j}, x_i + \Delta, x_j - \Delta) - V_i^*(x)| \\ &\quad \le \frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}. \end{aligned}$$

(v) *Function V_i^* is symmetric in the following sense*: *for any two $k, j \ne i$, $V_i^*(x_{-j-k}, x_k, x_j) = V_i^*(x)$.*

Remark 3 Again, the results for 2 players suggest that V^* has stronger properties, namely that it is nondecreasing in the following sense:

$$V_i^*(x_{-i-j}, x_i + \Delta, x_j - \Delta) - V_i^*(x) \ge 0 \quad \text{for } \Delta > 0.$$

The following example contradicts this supposition.

Take 3 symmetric players with $u_i(x) = x$, $M = 7$ and $\beta = 0.15$. One may numerically check that the profile $(\sigma_1, \sigma_2, \sigma_3)$ of simple bold selectors is an equilibrium in such a game. However: $V_1^*(0, 2, 5) = 0.0588$, $V_1^*(1, 2, 4) = 0.0574$, $V_1^*(2, 2, 3) = 0.0588$, $V_1^*(3, 2, 2) = 1.0588$, $V_1^*(4, 2, 1) = 1.0574$, $V_1^*(5, 2, 0) = 1.0588$, so V_1^* is not monotonic in the above sense. We suppose that the lower bound in part (iv) of Theorem 2 could however be improved upon.

3.1 Analysis of the Results

The results from Theorem 1 can be read in the following manner: as the number of players grows, it is rational to be more aggressive, but only to some extent. If we analyze the proof of this theorem, we see that the main reason for which the players become more aggressive is that after a player loses his position as one of the leaders in terms of amount of money held, it is hard for him to return to a leading position, and as n grows, the probability of returning to one of these top positions becomes smaller. As our model is based on auctions in subsequent steps of the game, only the money leaders can actively participate in the trade. Now, having in mind the main dilemma of a player in our setting, which is whether to spend more now or to save it for the future consumption, we see that if a player does not bid high when he has a possibility of winning an auction, he risks that he will lose this possibility for a long time. If he does, he may be in a better strategic situation when he is back in play, but after such a long time his rewards will be discounted heavily, so he will not be able to take advantage of this better strategic situation.

This also explains why it is sometimes better to be a little bit less aggressive (as it is the case when weakly bold strategies are applied). Note that what we just wrote can be put in another way as follows: What counts for a player is what he can gain up to the moment when he loses position as one of the money leaders. That means that it is rational for a player when n is high enough, to chose his strategy as if the game was finished at the moment when he loses his position. In a particular situation appearing in the proof of part (iii) of Theorem 1, a player knows that bidding less at the beginning of the game will lower his expected daily reward by half, but at the same time it will add him one more stage in which he will be an active trader, so for a relatively high value of β saving this half for the future consumption will be rational.

This mechanism of choosing the strategy which would be optimal in a game with finitely many periods seems to be the core of all the other anomalies which are enumerated in remarks after Theorems 1 and 2. If at some stage bidding more than what is needed to win an auction (and what is prescribed by the simple bold strategy) gives (at least with some probability) a possibility to add one more stage of activity, it is better to use this more aggressive strategy. If a state with a bigger wealth is strategically worse (in the sense described above) than in another state with a smaller wealth, the reward of a player in this first one is smaller than in the second one.

3.2 Possible Extensions

There are three main directions in which one could think of extending the results presented in this paper. The first one concerns the rule of distribution of the good based on the bids. Two standard rules appearing in the literature are auction and proportional distribution. Obviously one could imagine some more complicated rules, and, as long as in a rule of distribution of the good only a small group of the highest bidders can gain anything (and small should mean shrinking in terms of percentage as n grows), some counterpart of the results presented here (the asymptotic result in particular) should definitely be true.

The second possible extension of our results is generalizing them to some other rules of returning the money to the players after the consumption. It is clear from our considerations that the same kind of result will hold if the probability of getting back in trade after losing such a possibility decreases to 0 as n goes to infinity. So the main point here would be checking if for a given rule different than the ours this is true.

The third interesting direction in which the results of our model could be generalized is allowing borrowing and lending, as in [5] for a game with continuum of players. We think that in such a case our results could not be repeated, as the possibility of lending would mitigate the effect of losing the position, unless the interest rates were relatively high—then a result similar to ours seems highly possible.

4 Proofs

This section contains the proofs of all the theorems included in previous section. The proof of the corollary is omitted as straightforward.

4.1 Proof of Theorem 1

We precede main part of the proof by two technical lemmas. In the first one relations between different constants appearing in the theorem's proof are established.

Lemma 1 *For any $n \in \mathbb{N}$ and $\beta \in (0, 1)$:*

(i)

$$\begin{aligned}\frac{\beta u_i(1)}{(n-(n-1)\beta)(1-\beta)} &< \frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)} \\ &< \frac{n u_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)} \qquad (1)\end{aligned}$$

(ii)

$$\begin{aligned}& u_i(1) + \frac{(n-2)\beta}{n}\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)} \\ &\quad + \frac{2\beta}{n}\left(\frac{n u_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right) \\ &= \frac{n u_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)} \\ &> \frac{(n-1)\beta}{n}\left(\frac{n u_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right) + \frac{\beta u_1(1)}{n(1-\beta)}.\end{aligned}$$

Proof (i) To prove the first inequality first notice that $\beta < 1$ implies

$$n-(n-2)\beta < 2\big(n-(n-1)\beta\big).$$

But this immediately implies that

$$\frac{\beta u_i(1)}{(n-(n-1)\beta)(1-\beta)} < \frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)}.$$

The second inequality follows also from $\beta < 1$. Note that it implies that

$$2\beta < n-(n-2)\beta$$

and consequently

$$\frac{2\beta}{n-(n-2)\beta} < 1.$$

We can further transform this inequality as follows:

$$n(1-\beta)\frac{2\beta}{(n-(n-2)\beta)(1-\beta)} < n,$$

$$\big((n-2\beta)-\beta(n-2)\big)\frac{2\beta}{(n-(n-2)\beta)(1-\beta)} < n,$$

$$\frac{n-2\beta}{n-2\beta}\frac{2\beta}{(n-(n-2)\beta)(1-\beta)} - \frac{\beta(n-2)}{n-2\beta}\frac{2\beta}{(n-(n-2)\beta)(1-\beta)} < \frac{n}{n-2\beta},$$

and finally:

$$\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)} < \frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}.$$

(ii) $u_i(1) + \frac{(n-2)\beta}{n}\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)} + \frac{2\beta}{n}(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)})$ equals

$$\begin{aligned}&\frac{n-2\beta}{n}\left(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\\ &\quad + \frac{2\beta}{n}\left(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\\ &= \frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}.\end{aligned}$$

To prove the inequality note that the inequality between two utmost terms in (1) can equivalently be written as

$$\frac{n-(n-1)\beta}{n}\left(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right) > \frac{\beta u_i(1)}{n(1-\beta)}.$$

This implies that

$$\begin{aligned}&\frac{(n-1)\beta}{n}\left(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\\ &< \left(\frac{(n-1)\beta}{n} + \frac{n-(n-1)\beta}{n}\right)\left(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\\ &= \frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}.\end{aligned}$$

□

The second lemma is straightforward, but we write it for the sake of the paper's completeness.

Lemma 2 *For every i, F_i^{WB} is a compact convex subset of $\mathbb{R}^{M|S|}$.*

Now we can make the first step of the actual proof of the theorem. Fix $n > 2$ and for any profile of stationary strategies f define the operator $T_{f,i} : \mathcal{B}(S, \mathbb{R}_+) \to \mathcal{B}(S, \mathbb{R}_+)$ by the formula:

$$T_{f,i}(v)(x) = E_{x,f}\big[r_i\big(x, f(x)\big) + \beta v\big(x^2\big)\big]. \tag{2}$$

It is well-known from the theory of dynamic programming (see e.g. [7, 8]) that $T_{f,i}(v)(x)$ represents the expected payoff to player i when he plays $f_i(x)$ versus $f_{-i}(x)$ on the first day, receives his one-day reward and is paid $v(x^2)$ at the next state x^2. It is also well-known that $T_{f,i}$ is a contraction operator.

Next we can define the reward function corresponding to a best (maximizing the payoff of player i) weakly bold strategy of player i, say f_i (such a strategy exists, as by Lemma 2 F_i^{WB} is a compact subset of a Euclidean space and discounted reward is a continuous function of the strategy (by some standard dynamic programming arguments, see e.g. Lemma 2.1 in [1])), when all the other players use some fixed symmetric profile of weakly bold strategies[4] $f_{-i} = (f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n)$, $V_i^{f_{-i}} : S \to \mathbb{R}_+$. Again by standard dynamic programming arguments, $V_i^{f_{-i}}$ is the unique fixed point of $T_{f,i}$.

Next we define the set $\mathcal{V}_i$ of functions satisfying the five conditions appearing in Theorem 2:

$$\begin{aligned}
\mathcal{V}_i = \Bigg\{ & v : S \to \mathbb{R}_+ : \forall x \in S,\ v(x) \le \frac{u_i(1)}{1-\beta};\\
& \forall x \in S \text{ such that } x_i < x_{(2)},\ v(x) \le \frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)};\\
& \forall x \in S \text{ such that } x_i < 2x_{(2)},\\
& v(x) \le \frac{n u_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)};\\
& \forall x \in S \text{ such that } x_i = x_{(1)}, k \text{ such that } x_k = x_{(2)},\ j \ne i, k \text{ and } \Delta \le x_{(2)},\\
& |v(x_{-i-j}, x_i + \Delta, x_j - \Delta) - v(x)|\\
& \quad \le \frac{n u_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)};\\
& \forall j, k \ne i,\ v(x_{-j-k}, x_k, x_j) = v(x) \Bigg\}.
\end{aligned}$$

Clearly it is a closed subset of a complete metric space of bounded functions from S to $\mathbb{R}_+$ with supremum norm, $\mathcal{B}(S, \mathbb{R}_+)$, and hence also a complete metric space. We can prove the following:

[4] Strategy f_i is obviously induced by f_{-i}. We give it a name and justify its existence because we shall use it in the proof of Lemma 3.

Lemma 3 *Let $\beta \in (0,1)$. Then $V_i^{f_{-i}}$ is a member of $\mathcal{V}_i$.*

Proof By the definition of weakly bold strategies, in our case (2) can be rewritten in the following form:

$$
\begin{aligned}
&T_{f,i}(v)(x)\\
&=\begin{cases}
\sum_{a=x_{(2)}+1}^{x_i}(f_i(x))_a[u_i(1)+\frac{\beta}{n}\sum_{k\neq i}v(x_{-i-k},x_i-a,x_k+a)+\frac{\beta}{n}v(x)]\\
\quad+\frac{(f_i(x))_{x_{(2)}}}{|J_{(2)}(x)|}[u_i(1)\\
\quad+\sum_{j\in J_{(2)}(x)}(\frac{\beta}{n}\sum_{k\neq j}v(x_{-j-k},x_j-x_{(2)},x_k+x_{(2)})+\frac{\beta}{n}v(x))]\\
\quad\text{if } i\in J_{(1)}(x),\\
\sum_{a=x_{(2)}+1}^{x_{j_1}}(f_{j_1}(x))_a[\frac{\beta}{n}\sum_{k\neq j_1}v(x_{-j_1-k},x_{j_1}-a,x_k+a)+\frac{\beta}{n}v(x)]\\
\quad+\frac{(f_{j_1}(x))_{x_{(2)}}}{|J_{(2)}(x)|}[u_i(1)\\
\quad+\sum_{j\in J_{(2)}(x)}(\frac{\beta}{n}\sum_{k\neq j}v(x_{-j-k},x_j-x_{(2)},x_k+x_{(2)})+\frac{\beta}{n}v(x))]\\
\quad\text{if } i\in J_{(2)}(x)\setminus J_{(1)}(x) \text{ and } j_1\in J_{(1)}(x),\\
\sum_{a=x_{(2)}+1}^{x_{j_1}}(f_{j_1}(x))_a[\frac{\beta}{n}\sum_{k\neq j_1}v(x_{-j_1-k},x_{j_1}-a,x_k+a)+\frac{\beta}{n}v(x)]\\
\quad+\frac{(f_{j_1}(x))_{x_{(2)}}}{|J_{(2)}(x)|}[\sum_{j\in J_{(2)}(x)}(\frac{\beta}{n}\sum_{k\neq j}v(x_{-j-k},x_j-x_{(2)},x_k+x_{(2)})\\
\quad+\frac{\beta}{n}v(x))]\\
\quad\text{if } i\notin J_{(2)}(x) \text{ and } j_1\in J_{(1)}(x),
\end{cases}
\end{aligned}
\tag{3}
$$

where $J_{(1)}(x)=\{j : x_j = x_{(1)}\}$ and $J_{(2)}(x)=\{j : x_j \geq x_{(2)}\}$. (Note that j_1 can be chosen arbitrarily in case when $|J_{(1)}(x)|>1$, since in that case $J_{(1)}(x)=J_{(2)}(x)$ and the first term when $i\notin J_{(2)}(x)$ disappears).

As we already know, $T_{f,i}$ is a contraction. In the remainder of the proof we will show that $T_{f,i}$ maps $\mathcal{V}_i$ into $\mathcal{V}_i$. This of course will imply through Banach's fixed point theorem, that the fixed point of $T_{f,i}$, $V_i^{f_{-i}}$, is an element of $\mathcal{V}_i$, ending the proof. We will break this part of the proof into five claims concerning each element of the definition of $\mathcal{V}_i$. Suppose $v\in\mathcal{V}_i$.

Claim 1 $T_{f,i}(v)(x)\leq\frac{u_i(1)}{1-\beta}$.

Clearly for any $x\in S$:

$$T_{f,i}(v)(x)\leq u_i(1)+\beta\max_{x'\in S}v\big(x'\big)\leq u_i(1)+\frac{u_i(1)\beta}{1-\beta}=\frac{u_i(1)}{1-\beta}.$$

Claim 2 *For every $x\in S$ such that $x_i<x_{(2)}$, $T_{f,i}(v)(x)\leq\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)}$.*

First note that

$$
\begin{aligned}
&T_{f,i}(v)(x)\\
&= \sum_{a=x_{(2)}+1}^{x_{j_1}} \big(f_{j_1}(x)\big)_a \left[\frac{\beta}{n}\sum_{k\neq j_1} v(x_{-j_1-k}, x_{j_1}-a, x_k+a) + \frac{\beta}{n}v(x)\right]\\
&\quad + \frac{(f_{j_1}(x))_{x_{(2)}}}{|J_{(2)}(x)|}\left[\sum_{j\in J_{(2)}(x)}\left(\frac{\beta}{n}\sum_{k\neq j} v(x_{-j-k}, x_j - x_{(2)}, x_k + x_{(2)}) + \frac{\beta}{n}v(x)\right)\right]\\
&= \frac{\beta}{n}\Bigg\{\sum_{a=x_{(2)}+1}^{x_{j_1}} \big(f_{j_1}(x)\big)_a v(x_{-j_1-i}, x_{j_1}-a, x_i+a)\\
&\quad + \frac{(f_{j_1}(x))_{x_{(2)}}}{|J_{(2)}(x)|}\sum_{j\in J_{(2)}(x)} v(x_{-j-i}, x_j - x_{(2)}, x_i + x_{(2)})\Bigg\}\\
&\quad + \frac{\beta}{n}\Bigg\{\sum_{a=x_{(2)}+1}^{x_{j_1}} \big(f_{j_1}(x)\big)_a \sum_{k\in J_{(2)}(x)\setminus\{j_1\}} v(x_{-j_1-k}, x_{j_1}-a, x_k+a)\\
&\quad + \frac{(f_{j_1}(x))_{x_{(2)}}}{|J_{(2)}(x)|}\sum_{j\in J_{(2)}(x)}\sum_{k\in J_{(2)}(x)\setminus\{j\}} v(x_{-j-k}, x_j - x_{(2)}, x_k + x_{(2)})\Bigg\}\\
&\quad + \frac{\beta}{n}\Bigg\{\sum_{a=x_{(2)}+1}^{x_{j_1}} \big(f_{j_1}(x)\big)_a \left[\sum_{k\notin J_{(2)}(x)\cup\{i\}} v(x_{-j_1-k}, x_{j_1}-a, x_k+a) + v(x)\right]\\
&\quad + \frac{(f_{j_1}(x))_{x_{(2)}}}{|J_{(2)}(x)|}\left[\sum_{j\in J_{(2)}(x)}\left(\sum_{k\notin J_{(2)}(x)\cup\{i\}} v(x_{-j-k}, x_j - x_{(2)}, x_k + x_{(2)}) + v(x)\right)\right]\Bigg\}.
\end{aligned}
$$

Now note that the values of v can be bigger than $\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)}$ only in the first two brackets. In the first one this is, because the wealth of player i is increased and so he may enter $J_{(2)}(x)$. In the second one it is, because whenever some amount of money is moved between two players from $J_{(2)}(x)$, the wealth of one of them can become smaller than that of player i, and so i may enter $J_{(2)}(x)$ in his place. Note however that this is only possible when $|J_{(2)}(x)| = 2$—in any other case only the values of v in the first bracket can be bigger than $\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)}$. Putting together all that we wrote, $T_{f,i}(v)(x)$ cannot be bigger than

$$
\begin{aligned}
&\frac{\beta}{n}\left[\sum_{a=x_{(2)}+1}^{x_{j_1}} \big(f_{j_1}(x)\big)_a \frac{u_i(1)}{1-\beta} + \frac{(f_{j_1}(x))_{x_{(2)}}}{|J_{(2)}(x)|}|J_{(2)}(x)|\frac{u_i(1)}{1-\beta}\right]\\
&\quad + \frac{\beta}{n}\left[\sum_{a=x_{(2)}+1}^{x_{j_1}} \big(f_{j_1}(x)\big)_a \frac{u_i(1)}{1-\beta} + \frac{(f_{j_1}(x))_{x_{(2)}}}{|J_{(2)}(x)|}|J_{(2)}(x)|\frac{u_i(1)}{1-\beta}\right]
\end{aligned}
$$

$$+\frac{\beta}{n}\Bigg[\sum_{a=x_{(2)}+1}^{x_{j_1}}\big(f_{j_1}(x)\big)_a\big(n-|J_{(2)}(x)|\big)\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)}$$

$$+\frac{(f_{j_1}(x))_{x_{(2)}}}{|J_{(2)}(x)|}|J_{(2)}(x)|\big(n-|J_{(2)}(x)|\big)\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)}\Bigg]$$

$$=\frac{\beta}{n}\frac{u_i(1)}{1-\beta}+\frac{\beta}{n}\frac{u_i(1)}{1-\beta}+\frac{\beta}{n}\big(n-|J_{(2)}(x)|\big)\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)}$$

$$\leq\frac{2\beta u_i(1)}{n(1-\beta)}+\frac{2\beta^2(n-2)u_i(1)}{n(n-(n-2)\beta)(1-\beta)}=\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)},$$

ending the proof of Claim 2.

Claim 3 *For every $x \in S$ such that $x_i < 2x_{(2)}$,*

$$T_{f,i}(v)(x)\leq\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}.$$

Clearly we only need to consider the case when $i \in J_{(2)}(x)$ (the possibility when $i \notin J_{(2)}(x)$ is, as a consequence of part (i) of Lemma 1, covered by Claim 2). As we use the characterization (3) of $T_{f,i}$ which has three variants, it is convenient to consider two cases:

Case 1: $i \in J_{(1)}(x)$. In this case we can write $T_{f,i}(v)(x)$ as follows:

$$\sum_{a=x_{(2)}+1}^{x_i}\big(f_i(x)\big)_a\Bigg[u_i(1)+\frac{\beta}{n}\sum_{k\neq i}v(x_{-i-k},x_i-a,x_k+a)+\frac{\beta}{n}v(x)\Bigg]$$

$$+\frac{(f_i(x))_{x_{(2)}}}{|J_{(2)}(x)|}\sum_{j\in J_{(2)}(x)\setminus\{i\}}\left(\frac{\beta}{n}\sum_{k\neq j}v(x_{-j-k},x_j-x_{(2)},x_k+x_{(2)})+\frac{\beta}{n}v(x)\right)$$

$$+\frac{(f_i(x))_{x_{(2)}}}{|J_{(2)}(x)|}\Bigg[u_i(1)+\left(\frac{\beta}{n}\sum_{k\neq i}v(x_{-i-k},x_j-x_{(2)},x_k+x_{(2)})+\frac{\beta}{n}v(x)\right)\Bigg]$$

$$\leq\sum_{a=x_{(2)}+1}^{x_i}\big(f_i(x)\big)_a\Bigg[u_i(1)+\frac{(n-2)\beta}{n}\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)}$$

$$+\frac{2\beta}{n}\left(\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\Bigg]$$

$$+\frac{(f_i(x))_{x_{(2)}}}{|J_{(2)}(x)|}\frac{(|J_{(2)}|-1)\beta}{n}$$

$$\times\Bigg[(n-1)\left(\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)+\frac{u_1(1)}{1-\beta}\Bigg]$$

$$
\begin{aligned}
&+ \frac{(f_i(x))_{x_{(2)}}}{|J_{(2)}(x)|}\bigg[u_i(1) + \frac{(n-2)\beta}{n}\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)} \\
&+ \frac{2\beta}{n}\bigg(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\bigg)\bigg] \\
\leq \max\bigg\{ &u_i(1) + \frac{(n-2)\beta}{n}\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)} \\
&+ \frac{2\beta}{n}\bigg(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\bigg), \\
&\frac{(n-1)\beta}{n}\bigg(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\bigg) + \frac{\beta u_1(1)}{n(1-\beta)}\bigg\}.
\end{aligned}
$$

In the above set of inequalities the first one is a consequence of the following two facts:

(1) In all of the states of the form[5] $y = (x_{-j-k}, x_j - a, x_k + a)$ where $a \geq x_{(2)}$ and $j, k \neq i$, either $y_i = y_{(2)}$ or $i, k \in J_{(2)}(y)$ with $y_i = x_i = y_{(1)}$ and $y_k = y_{(2)} \geq x_{(2)}$. In both cases obviously $y_i \leq 2y_{(2)}$. Only when $k = i$, it is possible that $y_i \not\leq 2y_{(2)}$.

(2) In the states $z = (x_{-i-k}, x_i - a, x_k + a)$ where $a \geq x_{(2)}$ and $k \neq i$, clearly $z_i < x_{(2)} \leq z_k$. If in addition $k \notin J_{(2)}(x)$ or $|J_{(2)}(x)| > 2$, there exists some $j \neq i, k$ such that $z_j = x_j = x_{(2)} > z_i$, and so for any $k \neq i, j$, $z \notin J_{(2)}(z)$.

The second inequality is straightforward, as f_i is a randomized stationary strategy, and so the $(f_i(x))_a$ for different values of a sum up to 1.

Finally, by part (ii) of Lemma 1 both terms under *max* are not bigger than

$$
\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)},
$$

which ends the proof of Case 1.

Case 2: $i \in J_{(2)}(x) \setminus J_{(1)}(x)$. In this case $T_{f,i}(v)(x)$ is (for some $j_1 \in J_{(1)}$) equal to:

$$
\begin{aligned}
&\sum_{a=x_{(2)}+1}^{x_{j_1}} \big(f_{j_1}(x)\big)_a \bigg[\frac{\beta}{n}\sum_{k\neq j_1} v(x_{-j_1-k}, x_{j_1} - a, x_k + a) + \frac{\beta}{n}v(x)\bigg] \\
&\quad + \frac{(f_{j_1}(x))_{x_{(2)}}}{|J_{(2)}(x)|}\sum_{j\in J_{(2)}(x)\setminus\{i\}}\bigg(\frac{\beta}{n}\sum_{k\neq j} v(x_{-j-k}, x_j - x_{(2)}, x_k + x_{(2)}) + \frac{\beta}{n}v(x)\bigg) \\
&\quad + \frac{(f_{j_1}(x))_{x_{(2)}}}{|J_{(2)}(x)|}\bigg[u_i(1) + \bigg(\frac{\beta}{n}\sum_{k\neq i} v(x_{-i-k}, x_j - x_{(2)}, x_k + x_{(2)}) + \frac{\beta}{n}v(x)\bigg)\bigg].
\end{aligned}
$$

[5] If we wanted to be precise, we should write "in state $y^{jka} = \cdots$", however adding this kind of information complicates the notation without adding anything important for understanding this explanation. We use similar convention in other explanations in the remainder of the paper.

Now we can repeat the considerations of Case 1 to show that this is not bigger than:

$$\begin{aligned}
&\sum_{a=x_{(2)}+1}^{x_{j_1}} \big(f_{j_1}(x)\big)_a \left[(n-1)\left(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right) + \frac{u_1(1)}{1-\beta}\right] \\
&\quad + \frac{(f_{j_1}(x))_{x_{(2)}}}{|J_{(2)}(x)|} \frac{(|J_{(2)}|-1)\beta}{n} \\
&\quad \times \left[(n-1)\left(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right) + \frac{u_1(1)}{1-\beta}\right] \\
&\quad + \frac{(f_{j_1}(x))_{x_{(2)}}}{|J_{(2)}(x)|} \left[u_i(1) + \frac{(n-2)\beta}{n} \frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)}\right. \\
&\quad \left. + \frac{2\beta}{n}\left(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\right] \\
&\le \frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)},
\end{aligned}$$

ending the proof of Case 2, and hence of whole Claim 3.

Claim 4 *Suppose that $\Delta \le x_{(2)}$ and that $x_i = x_{(1)}$, $x_k = x_{(2)}$ and $j \ne i, k$. Then*

$$\begin{aligned}
&|T_{f,i}(v)(x_{-i-j}, x_i+\Delta, x_j-\Delta) - T_{f,i}(v)(x)| \\
&\quad \le \frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}.
\end{aligned}$$

We will only consider the case when $x_i + \Delta \ge 2x_k$ (if $x_i + \Delta < 2x_k$, the result follows immediately from Claim 3 and the fact that $T_{f,i}(v) \ge 0$). To make the notation (which is quite complicated anyway) shorter, we shall write y instead of $(x_{-i-j}, x_i + \Delta, x_j - \Delta)$. We will only prove

$$T_{f,i}(v)(y) - T_{f,i}(v)(x) \le \frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}. \tag{4}$$

The proof of the other inequality (with $T_{f,i}(v)(x) - T_{f,i}(v)(y)$ on the LHS of the inequality) follows along the same lines, with only difference that we need to modify the strategy in state y, instead of in x.

Define $\tilde{f}_i$ to be the strategy f_i modified (only) in state x in the following manner[6]:

$$\begin{aligned}
\big(\tilde{f}_i(x)\big)_a &= \big(f_i(y)\big)_a \quad \text{if } x_k < a \le x_i, \\
\big(\tilde{f}_i(x)\big)_{x_k} &= \big(f_i(y)\big)_{x_k} + \sum_{a > x_i} \big(f_i(y)\big)_a.
\end{aligned}$$

[6]Consequently we will also write $\tilde{f}$ for the whole profile f with f_i replaced by $\tilde{f}_i$.

Note that $\tilde{f}_i$ is still a weakly bold strategy. We will show that

$$T_{f,i}(v)(y) - T_{\tilde{f},i}(v)(x) \le \frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}.$$

Since by the definition f_i is a best weakly bold strategy of player i, $T_{\tilde{f},i}(v)(x) \le T_{f,i}(v)(x)$, which will immediately imply (4).

By (3) applied to our situation we have:

$$\begin{aligned}
&T_{f,i}(v)(y) - T_{\tilde{f},i}(v)(x)\\
&= \Bigg\{ \sum_{a=x_k+1}^{x_i+\Delta} \big(f_i(y)\big)_a \Bigg[u_i(1) + \frac{\beta}{n}\sum_{l\neq i} v(y_{-i-l}, y_i - a, y_l + a) + \frac{\beta}{n} v(y)\Bigg]\\
&\quad + \frac{(f_i(y))_{x_k}}{|J_{(2)}(y)|}\Bigg[u_i(1) + \frac{\beta}{n}\sum_{l\neq i} v(y_{-i-l}, y_i - x_k, y_l + x_k) + \frac{\beta}{n} v(y)\Bigg]\\
&\quad + \frac{(f_i(y))_{x_k}}{|J_{(2)}(y)|}\sum_{m\in J_{(2)}(y)\setminus\{i\}}\Bigg[\frac{\beta}{n}\sum_{l\neq m} v(y_{-m-l}, y_m - x_k, y_l + x_k) + \frac{\beta}{n} v(y)\Bigg]\Bigg\}\\
&\quad - \Bigg\{ \sum_{a=x_k+1}^{x_i} \big(f_i(y)\big)_a \Bigg[u_i(1) + \frac{\beta}{n}\sum_{l\neq i} v(x_{-i-l}, x_i - a, x_l + a) + \frac{\beta}{n} v(x)\Bigg]\\
&\quad + \frac{(f_i(y))_{x_k} + \sum_{a>x_i}(f_i(y))_a}{|J_{(2)}(x)|}\\
&\quad \times \Bigg[u_i(1) + \frac{\beta}{n}\sum_{l\neq i} v(x_{-i-l}, x_i - x_k, x_l + x_k) + \frac{\beta}{n} v(y)\Bigg]\\
&\quad + \frac{(f_i(y))_{x_k} + \sum_{a>x_i}(f_i(y))_a}{|J_{(2)}(x)|}\\
&\quad \times \sum_{m\in J_{(2)}(x)\setminus\{i\}}\Bigg[\frac{\beta}{n}\sum_{l\neq m} v(x_{-m-l}, x_m - x_k, x_l + x_k) + \frac{\beta}{n} v(x)\Bigg]\Bigg\}.
\end{aligned} \tag{5}$$

Now note that $J_{(2)}(y) \subset J_{(2)}(x)$ and the only difference between the two sets may be j, which may be an element of $J_{(2)}(x)$ and not of $J_{(2)}(y)$. Thus we have two cases:

Case 1: $J_{(2)}(x) = J_{(2)}(y)$. We can omit all the elements in the second bracket of (5) which are multiplied by $\frac{\sum_{a>x_i}(f_i(y))_a}{|J_{(2)}(x)|}$ (as they are all positive), obtaining that in this case (5) is not bigger than:

$$\begin{aligned}
&\sum_{a=x_i+1}^{x_i+\Delta} \big(f_i(y)\big)_a \Bigg[u_i(1) + \frac{\beta}{n}\sum_{l\neq i} v(y_{-i-l}, y_i - a, y_l + a) + \frac{\beta}{n} v(y)\Bigg]\\
&\qquad + \sum_{a=x_k+1}^{x_i} \big(f_i(y)\big)_a \Bigg[\frac{\beta}{n}\sum_{l\neq i}\big(v(y_{-i-l}, y_i - a, y_l + a)
\end{aligned}$$

$$- v(x_{-i-l}, x_i - a, x_l + a)\big) + \frac{\beta}{n}\big(v(y) - v(x)\big)\Bigg]$$

$$+ \frac{(f_i(y))_{x_k}}{|J_{(2)}(y)|}\Bigg[\frac{\beta}{n}\sum_{l\neq i}\big(v(y_{-i-l}, y_i - x_k, y_l + x_k)$$

$$- v(x_{-i-l}, x_i - x_k, x_l + x_k)\big) + \frac{\beta}{n}\big(v(y) - v(x)\big)\Bigg]$$

$$+ \frac{(f_i(y))_{x_k}}{|J_{(2)}(y)|}\sum_{m\in J_{(2)}(y)\setminus\{i\}}\Bigg[\frac{\beta}{n}\sum_{l\neq m}\big(v(y_{-m-l}, y_m - x_k, y_l + x_k)$$

$$- v(x_{-m-l}, x_m - x_k, x_l + x_k)\big) + \frac{\beta}{n}\big(v(y) - v(x)\big)\Bigg]. \qquad (6)$$

Now note that the value in the first bracket cannot be bigger than $u_i(1) + \frac{(n-2)\beta}{n}\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)} + \frac{2\beta}{n}(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)})$, as if $a > x_i$, then the fortune of player i in state $z = (y_{-i-l}, y_i - a, y_l + a)$ would be smaller than Δ, and thus, as long as $l \neq k$, also $z_i < x_k = z_k$ and $z_i < z_l$, so $i \notin J_{(2)}(z)$. Of course when $k = l$, $z_i < 2z_{(2)}$, as either $i \notin J_{(2)}(z)$ or $z_i = z_{(2)}$ then.

The next thing we need to note is that the values in the second and the third bracket can be bounded from above by $\beta(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)})$. This is because in this case we have one of two situations:

(1) For the state $s = (x_{-i-l}, x_i - x_k, x_l + x_k)$, i is the leader, and then, as $s_l = x_l + x_k > x_k \geq \Delta$, we can use the assumption that $v(s_{-i-j}, s_i + \Delta, s_j + \Delta) - v(s) \leq \frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}$.

(2) i is not the leader in state s, and then we clearly see that

$$y_i - x_k = s_i + \Delta < s_l + \Delta = (x_l + x_k) + \Delta = (y_l + x_k) + \Delta,$$

so in state $s^\Delta = (y_{-i-l}, y_i - x_k, y_l + x_k)$, $s_i^\Delta < s_l^\Delta + \Delta \leq 2s_l^\Delta$, and thus $v(s^\Delta) - v(s) \leq v(s^\Delta) \leq \frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}$.

Finally, the values in the fourth bracket are not bigger than $\frac{(n-1)\beta}{n}(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}) + \frac{\beta u_1(1)}{n(1-\beta)}$. Here the situation is analogous to that in the second and the third brackets, only when $l = i$, the difference in the bracket can be bigger than $\beta(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)})$ (as here we do not know who enters $J_{(2)}$ in place of player k), but not bigger than $\frac{u_i(1)}{1-\beta}$, which is the biggest possible value of v.

Since the coefficients of these four brackets are all positive and sum up to 1, (6) cannot be bigger than

$$\max\Bigg\{u_i(1) + \frac{(n-2)\beta}{n}\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)}$$

$$+ \frac{2\beta}{n}\left(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right),$$

$$\beta\left(\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right),$$
$$\left.\frac{(n-1)\beta}{n}\left(\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)+\frac{\beta u_1(1)}{n(1-\beta)}\right\}$$
$$\leq\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)},$$

where the last inequality follows from part (ii) of Lemma 1.

Case 2: $J_{(2)}(x)=J_{(2)}(y)\cup\{j\}$. It is important to notice that in this case $x_j=x_k=x_{(2)}$. Again we can omit all the elements in the second bracket of (5) multiplied by $\frac{\sum_{a>x_i}(f_i(y))_a}{|J_{(2)}(x)|}$. In this case this implies that (5) is not bigger than:

$$\sum_{a=x_i+1}^{x_i+\Delta}\big(f_i(y)\big)_a\left[u_i(1)+\frac{\beta}{n}\sum_{l\neq i}v(y_{-i-l},y_i-a,y_l+a)+\frac{\beta}{n}v(y)\right]$$
$$+\sum_{a=x_k+1}^{x_i}\big(f_i(y)\big)_a\left[\frac{\beta}{n}\sum_{l\neq i}\big(v(y_{-i-l},y_i-a,y_l+a)\right.$$
$$\left.-v(x_{-i-l},x_i-a,x_l+a)\big)+\frac{\beta}{n}\big(v(y)-v(x)\big)\right]$$
$$+\frac{(f_i(y))_{x_k}}{|J_{(2)}(y)|+1}\left[\frac{\beta}{n}\sum_{l\neq i}\big(v(y_{-i-l},y_i-x_k,y_l+x_k)\right.$$
$$\left.-v(x_{-i-l},x_i-x_k,x_l+x_k)\big)+\frac{\beta}{n}\big(v(y)-v(x)\big)\right]$$
$$+\frac{(f_i(y))_{x_k}}{|J_{(2)}(y)|+1}\sum_{m\in J_{(2)}(y)\setminus\{i\}}\left[\frac{\beta}{n}\sum_{l\neq m}\big(v(y_{-m-l},y_m-x_k,y_l+x_k)\right.$$
$$\left.-v(x_{-m-l},x_m-x_k,x_l+x_k)\big)+\frac{\beta}{n}\big(v(y)-v(x)\big)\right]$$
$$+\frac{(f_i(y))_{x_k}}{(|J_{(2)}(y)|+1)|J_{(2)}(y)|}\left[\frac{\beta}{n}\sum_{l\neq i}\big(v(y_{-i-l},y_i-x_k,y_l+x_k)\right.$$
$$\left.-v(x_{-j-l},x_j-x_k,x_l+x_k)\big)+\frac{\beta}{n}\big(v(y)-v(x)\big)\right]$$
$$+\frac{(f_i(y))_{x_k}}{|J_{(2)}(y)|+1)|J_{(2)}(y)|}\sum_{m\in J_{(2)}(y)\setminus\{i\}}\left[\frac{\beta}{n}\sum_{l\neq m}\big(v(y_{-m-l},y_m-x_k,y_l+x_k)\right.$$
$$\left.-v(x_{-j-l},x_j-x_k,x_l+x_k)\big)+\frac{\beta}{n}\big(v(y)-v(x)\big)\right].\tag{7}$$

Now note that it has already been proved (in the considerations of Case 1) that the values in each of the first four brackets is not bigger than $\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}$. We will show that the same is true for the two remaining ones.

To prove the first of desired inequalities take $s^l=(y_{-i-l},y_i-x_k,y_l+x_k)$. One can easily compute that $(x_{-j-l},x_j-x_k,x_l+x_k)=(s^l_{-i-j},s^l_i+(x_k-\Delta),s^l_j-(x_k-\Delta))$. But as $x_k-\Delta<x_k+x_l$ and $x_k-\Delta<x_i-x_k+\Delta$ (this last inequality is implied by the assumption that we made at the beginning of the proof of Claim 4 that $x_i+\Delta\geq 2x_k$), clearly $x_k-\Delta<s^l_{(2)}$ and so

$$
\begin{aligned}
&\frac{\beta}{n}\Big[\sum_{l\neq i}\big[v(y_{-i-l},y_i-x_k,y_l+x_k)-v(x_{-j-l},x_j-x_k,x_l+x_k)\big]+\big[v(y)-v(x)\big]\Big]\\
&\quad=\frac{\beta}{n}\Big[\sum_{l\neq i}\big[v\big(s^l\big)-v\big(s^l_{-i-j},s^l_i+(x_k-\Delta),s^l_j-(x_k-\Delta)\big)\big]+\big[v(y)-v(x)\big]\Big]\\
&\quad\leq\frac{\beta}{n}\Big[(n-1)\Big(\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\Big)\\
&\qquad+\Big(\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\Big)\Big]\\
&\quad<\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}.
\end{aligned}
$$

To prove the last inequality we need to use the assumption about symmetry of v and the fact that $x_j=x_m$ for each $m\in J_{(2)}(y)\setminus\{i\}$. If we take such an m:

$$
\begin{aligned}
&\frac{\beta}{n}\Big[\sum_{l\neq m}\big(v(y_{-m-l},y_m-x_k,y_l+x_k)-v(x_{-j-l},x_j-x_k,x_l+x_k)\big)+\big(v(y)-v(x)\big)\Big]\\
&\quad=\frac{\beta}{n}\Big[\sum_{l\neq m}\big(v(y_{-m-l},y_m-x_k,y_l+x_k)-v(x_{-m-l},x_m-x_k,x_l+x_k)\big)\\
&\qquad+\big(v(y)-v(x)\big)\Big]\\
&\quad\leq\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}.
\end{aligned}
$$

The inequality was proved in the proof of Case 1.

As we have shown that each of the brackets in (7) is not bigger than $\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}$, and the coefficients of each of them sum up to 1, this implies that also (7) is not bigger than $\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}$, ending the proof of Claim 4.

Claim 5 *For any $j.k \neq i$,*

$$T_{f,i}(v)(x_{-j-k}, x_k, x_j) = T_{f,i}(v)(x).$$

The claim follows immediately from the fact that v has the same property and the definition of $T_{f,i}$ (note that we do not use the actual indices of the players there, only index all the players form $J_{(1)}(x)$, $J_{(2)}(x)$ and all the ones outside these sets, making them interchangeable within these sets). □

In the remainder of the proof we shall consider the properties of one-step n-person game $\Gamma(f, x, \beta)$ defined by the following objects:

- The action set of player i is $A_i(x)$.
- The reward function of player i is defined for any action profile $a = (a_1, \dots, a_n)$ as

$$K_i^{f,x,\beta}(a) = \begin{cases} \frac{1}{|J_{(1)}(a)|}[u_i(1) \\ \quad + \frac{\beta}{n} \sum_{j\in J_{(1)}(a)} (\sum_{k\neq j} V_i^{f_{-i}}(x_{-j-k}, x_j - a_j, x_k + a_j) + V_i^{f_{-i}}(x))] \\ \quad \text{if } a_i \geq \max_{j\neq i}\{a_j\}, \\ \frac{1}{|J_{(1)}(a)|}[\frac{\beta}{n} \sum_{j\in J_{(1)}(a)} (\sum_{k\neq j} V_i^{f_{-i}}(x_{-j-k}, x_j - a_j, x_k + a_j) + V_i^{f_{-i}}(x))] \\ \quad \text{if } a_i < \max_{j\neq i}\{a_j\}. \end{cases}$$

Lemma 4 *Fix some symmetric profile of weakly bold strategies f and suppose all of the players except i use randomized strategies $f_j(x)$ in game $\Gamma(f, x, \beta)$. Then for every $\beta \leq 1 - \sqrt[3]{\frac{2n^2-6n+4}{n^3}}$ the set of best responses of player i against $f_{-i}(x)$ of the others in this game*

$$B_i^{f,x,\beta}(f_{-i}) := \left\{a_i \in A_i(x) : E^{f_{-i}}\left[K_i^{f,x,\beta}(a_{-i}, a_i)\right] = \max_{b_i\in A_i(x)} E^{f_{-i}}\left[K_i^{f,x,\beta}(a_{-i}, b_i)\right]\right\}$$

is either a subset of $\{x_{(2)}, \dots, x_i\}$ or, if this set is empty, is equal to $A_i(x)$.

Proof First note that clearly $K_i^{f,x,\beta}$ is constant on the set $\{a \in A_i(x) : a_i \leq x_{(2)} - 1\}$, as any weakly bold strategy of player j (f_j in particular) prescribes to use action x_j as long as j is not the leader, so either i is the leader and any action smaller than $x_{(2)} = \max_{j\neq i} x_j$ is smaller than what others use (in such a case he does not affect his own payoff), or, if i is not the leader, the leader, say player k, uses randomized strategy adopting only actions bigger than $x_{(2)} - 1$ (in such a case the action of player i would not affect his payoff either). This immediately implies that $B_i^{f,x,\beta}(f_{-i}) = A_i(x)$ when $x_i < x_{(2)}$. To complete the proof it is enough to show that for $i \in J_{(2)}(x)$

and $\beta \le 1 - \sqrt[3]{\frac{2n^2-6n+4}{n^3}}$,

$$E^{f_{-i}}\big[K_i^{f,x,\beta}(a_{-i}, x_{(2)})\big] > E^{f_{-i}}\big[K_i^{f,x,\beta}(a_{-i}, x_{(2)} - 1)\big].$$

Without loss of generality we may assume that all of the players use nonrandomized strategies and so we can skip the expected value operators, writing simply a_j for the action that $f_j(x)$ assigns probability 1.

Note that either $\max_{j \ne i} a_j > x_{(2)}$ and then $K_i^{f,x,\beta}(a_{-i}, x_{(2)}) = K_i^{f,x,\beta}(a_{-i}, x_{(2)} - 1)$ or

$$\begin{aligned}
&K_i^{f,x,\beta}(a_{-i}, x_{(2)}) - K_i^{f,x,\beta}(a_{-i}, x_{(2)} - 1) \\
&\quad = \frac{1}{|J_{(1)}(a_{-i})|(|J_{(1)}(a_{-i})| + 1)}\Bigg[|J_{(1)}(a_{-i})|u_i(1) + \frac{\beta}{n} \\
&\quad \times \sum_{j \in J_{(1)}(a_{-i})}\Bigg(V_i^{f_{-i}}(x_{-i-j}, x_i - x_{(2)}, x_j + x_{(2)}) \\
&\quad - V_i^{f_{-i}}(x_{-i-j}, x_i + x_{(2)}, x_j - x_{(2)}) \\
&\quad + \sum_{k \ne i,j}\big(V_i^{f_{-i}}(x_{-i-k}, x_i - x_{(2)}, x_k + x_{(2)}) \\
&\quad - V_i^{f_{-i}}(x_{-j-k}, x_j - x_{(2)}, x_k + x_{(2)})\big)\Bigg)\Bigg].
\end{aligned} \tag{8}$$

Since in each of the elements of the last sum $k \ne j$ and $j \in J_{(1)}(a_{-i})$ (or, in other words, $x_j = x_{(2)}$), we know that $x_{(2)} \le (x_{-i-k}, x_i - x_{(2)}, x_k + x_{(2)})_{(2)}$, as it is not bigger than the fortune of players j and k. We can thus use parts (i) and (iv) of Lemma 3 to obtain that (8) is not bigger than

$$\begin{aligned}
&\frac{1}{|J_{(1)}(a_{-i})| + 1}\Bigg[u_i(1) - \frac{\beta}{n}\Bigg((n-2)\Bigg(\frac{nu_i(1)}{n - 2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\Bigg) \\
&\qquad + \frac{u_i(1)}{1-\beta}\Bigg)\Bigg] \\
&\quad = \frac{u_i(1)}{|J_{(1)}(a_{-i})| + 1}\Bigg[\frac{n(n-2\beta)(n-(n-2)\beta)(1-\beta)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)} \\
&\qquad + \frac{-n(n-2)(n-(n-2)\beta)(1-\beta)\beta - 2(n-2)^2\beta^3 - (n-2\beta)(n-(n-2)\beta)\beta}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\Bigg] \\
&\quad = \frac{u_i(1)}{|J_{(1)}(a_{-i})| + 1}\,\frac{n^3(1-\beta)^3 - [(4-6n)\beta^3 + 3n^2\beta^2 - n^2\beta]}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}.
\end{aligned} \tag{9}$$

Now note that for any $n \geq 3$ the function $\phi^n(\beta) := (4-6n)\beta^3 + 3n^2\beta^2 - n^2\beta$ achieves its biggest value[7] on interval $[0,1]$ for $\beta = 1$, and so (note that the denumerator of (9), as well as coefficient depending on $u_i(1)$, are always positive) (9) is for any $\beta \in (0,1)$ bigger than

$$\frac{u_i(1)}{|J_{(1)}(a_{-i})|+1}\frac{n^3(1-\beta)^3-(2n^2-6n+4)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}.$$

Simple computations show that this is not smaller than 0 iff $\beta \leq 1 - \sqrt[3]{\frac{2n^2-6n+4}{n^3}}$, ending the proof. □

Now we are ready to pass to the main part of the proof of Theorem 1.

Proof of (ii) of Theorem 1 Let us construct the correspondence $B^\beta : F_i^{WB} \to 2^{F_i^{WB}}$:

$$B^\beta(f_i) = \{g_i \in F_i^{WB} : \forall x \in S, \operatorname{supp}(g_i(x)) \in B_i^{(\operatorname{sym}(f_i), f_i), x, \beta}(f_{-i})\}.$$

By Lemma 4 for every $\beta \leq 1 - \sqrt[3]{\frac{2n^2-6n+4}{n^3}}$, B^β is nonempty valued. It is immediate to see that all the values of B^β are convex and compact. Next note that by Lemma 2.1 in [1] the expected discounted reward of player i is for any given x a jointly continuous function of the strategies of the players. As the set F_i^{WB} is compact, also $V_i^{f_{-i}} = \max_{f_i \in F_i^{WB}} J_i(x,(f_{-i},f_i))$ is a continuous function of f_{-i}. This implies that clearly also for any $a_i \in A_i(x)$, $E^{f_{-i}}[K_i^{f,x,\beta}(a_{-i},a_i)]$ is a continuous function of f_{-i}. Therefore if we take sequences $f_i^n \to f_i$ and $g_i^n \to g_i$, satisfying for every n $g_i^n \in B_i^{(\operatorname{sym}(f_i^n), f_i^n), x, \beta}(f_{-i}^n)$, then, since $\operatorname{supp}(g_i) \subset \operatorname{supp}(g_i^n)$ for n large enough (otherwise g_i could not be the limit of (g_i^n)), for any such n and any $a_i \in \operatorname{supp}(g_i)$ the following inequality is true:

$$E^{\operatorname{sym}(f_i^n)}[K_i^{(\operatorname{sym}(f_i^n), f_i^n), x, \beta}(a_{-i},a_i)] \geq E^{\operatorname{sym}(f_i^n)}[K_i^{(\operatorname{sym}(f_i^n), f_i^n), x, \beta}(a_{-i},b_i)]$$

for any $b_i \in A_i(x)$. Passing to the limit we obtain

$$E^{\operatorname{sym}(f_i)}[K_i^{(\operatorname{sym}(f_i), f_i), x, \beta}(a_{-i},a_i)] \geq E^{\operatorname{sym}(f_i)}[K_i^{(\operatorname{sym}(f_i), f_i), x, \beta}(a_{-i},b_i)],$$

which means that $g_i \in B_i^{(\operatorname{sym}(f_i), f_i), x, \beta}(f_{-i})$. This proves that the graph of B^β is closed. Since also, by Lemma 2, F_i^{WB} is a convex compact subset of a Euclidean space, B^β satisfies all the assumptions of Kakutani's fixed point theorem [3]. Hence there exists a fixed point f_i^* for the correspondence B^β. By standard dynamic programming arguments[8] f_i^*, is the best response of player i to the profile

[7] This can be easily checked analyzing the way the sign of the derivative of ϕ^n changes on the interval and comparing the values of ϕ^n at the ends of the interval.

[8] Note that by assumption $V_i^{f_{-i}^*}$ is the reward corresponding to the best weakly bold strategy of player i against f_{-i}^*, while by Lemma 4 f_i^* is the best weakly bold answer to f_{-i}^*. These two facts obviously

$f^*_{-i} = \mathrm{sym}(f^*_i)$ not only in game $\Gamma(f^*, x, \beta)$, but also in the stochastic game, and thus $f^* = (\mathrm{sym}(f^*_i), f^*_i)$ is a symmetric equilibrium in the stochastic game.

Proof of (i) Let $\beta \leq \frac{1}{3}$. It can be easily checked that for any value of $n \geq 3$, $1 - \sqrt[3]{\frac{2n^2-6n+4}{n^3}} > \frac{1}{3}$, thus, by (ii) proved above, the game has a weakly bold equilibrium $(f_1, \dots, f_n)$ for such a β. Suppose that there exists a state x such that $x_{(1)} = x_i > x_{(2)}$, but player i is prescribed to bid $x_{(2)}$ with some positive probability (without loss of generality we may assume this is with probability 1). His reward in state x is thus not bigger than

$$\frac{u_i(1)}{2} + \frac{\beta u_i(1)}{1-\beta} \leq \frac{u_i(1)}{2} + \frac{u_i(1)}{2} = u_i(1).$$

But this is smaller than what he could gain by playing $x_{(2)} + 1$—in that case he would gain $u_i(1)$ in the first step and at least $\frac{\beta u_i(1)}{2n}$ in the second one (as with probability $\frac{1}{n}$ he returns to x, where he gains at least another $\frac{u_i(1)}{2}$). Thus there is no state x such that $x_{(1)} = x_i > x_{(2)}$ such that player i is prescribed to bid $x_{(2)}$ with some positive probability and so f_i is a bold strategy.

Proof of (iii) Let us consider state $x^* = (\mathbf{0}_{-i-j}, 2s+1, s)$ (where $\mathbf{0}$ denotes the vector of n zeros) and denote:

(1) by w^* the expected reward of player i when he starts in state x^* and uses some bold strategy f_i over whole course of the game against a vector of bold strategies from the others f_{-i},

(2) by v^* the expected reward of the same player when he choses to play s in the first stage of the game and plays according to f_i in the next stages (with opponents applying the same bold strategies f_{-i} as in the first case).

First note that in case (2) the player i receives reward $u_i(1)$ with probability $\frac{1}{2}$ and then moves to:

- state x^* or another symmetric state $(\mathbf{0}_{-i-k}, 2s+1, s)$ with probability $\frac{1}{2}$;
- state $(\mathbf{0}_{-i-j-k}, s+1, s, s)$ (for some $k \neq i, j$) with probability $\frac{1}{2} - \frac{1}{n}$—in this state he gains at least additional $\beta u_1(1)$ (in the second stage);
- state $(\mathbf{0}_{-i}, 3s+1)$ with probability $\frac{1}{2n}$—in this state he also gains not less than $\beta u_i(1)$ in addition;
- state $(\mathbf{0}_{-i-j}, s+1, 2s)$ with probability $\frac{1}{2n}$.

Thus his reward satisfies:

$$v^* \geq \frac{u_i(1)}{2} + \beta\left(\frac{w^*}{2} + \frac{(n-1)u_i(1)}{n}\right). \tag{10}$$

Now, in case (1), when i uses a bold strategy, he receives reward $u_i(1)$ in the first stage with probability 1 and then moves to:

- state x^* with probability $\frac{1}{n}$;

imply that $V_i^{f^*_{-i}}$ satisfies the Bellman equations restricted to strategies from F_i^{WB}. However, since by Lemma 4, f^*_i also maximizes the payoff of player i in case the action sets are not restricted, this is in fact a globally best response of i against f^*_{-i} of the others.

- a state $(\mathbf{0}_{-i-j-k}, s-\alpha, s, s+1+\alpha)$ for some $k \neq i, j$ and $\alpha \geq 0$ with probability $\frac{n-2}{n}$;
- state $(\mathbf{0}_{-i-j}, s-\alpha, 2s+1+\alpha)$, where $\alpha \geq 0$, with probability $\frac{1}{n}$.

This, through Lemma 3, implies that

$$\begin{aligned} w^* \leq u_i(1) &+ \frac{(1-p_0)\beta}{n}\Bigg[w^* + (n-2)\left(\frac{2\beta u_i(1)}{(n-(n-2)\beta)(1-\beta)}\right) \\ &+ \left(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\Bigg] \\ &+ \frac{p_0\beta}{n}\Bigg[w^* + (n-2)w^{**} + \left(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\Bigg], \end{aligned} \tag{11}$$

where p_0 is the probability that $\alpha = 0$, and w^{**} is the reward of player i using f_i against f_{-i} of the others in state $x^{**} = (\mathbf{0}_{-i-j-k}, s, s, s+1)$. Next, using the equality from (ii) of Lemma 1 and writing $u_i(1)$ as $p_0u_i(1) + (1-p_0)u_i(1)$, (11) can be rewritten as follows:

$$\begin{aligned} w^* \leq &\frac{1-p_0}{n}\Bigg[\beta w^* + (n-\beta)\left(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\Bigg] \\ &+ \frac{p_0}{n}\Bigg[nu_i(1) + \beta(n-2)w^{**} \\ &+ \beta\left(w^* + \frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\Bigg]. \end{aligned} \tag{12}$$

Note that in this particular state the behavior of players using bold strategies is uniquely defined (players i and j play s, while k plays $s+1$) and the game moves either to state symmetric to x^{**} (with probability $\frac{n-2}{n}$) or to x^* (with probability $\frac{1}{n}$), or to $(\mathbf{0}_{-i-j}, s, 2s+1)$ (also with probability $\frac{1}{n}$). This implies that

$$w^{**} \leq \frac{\beta}{n}\left((n-2)w^{**} + w^* + \left(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\right)$$

or equivalently

$$w^{**} \leq \frac{\beta\left(w^* + \frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)}{n-(n-2)\beta}.$$

Substituting this into (12) we obtain:

$$\begin{aligned} w^* \leq &\frac{1-p_0}{n}\Bigg[\beta w^* + (n-\beta)\left(\frac{nu_i(1)}{n-2\beta} + \frac{2(n-2)\beta^2 u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\Bigg] \\ &+ \frac{p_0}{n}\Bigg[nu_i(1) + \left(\frac{(n-2)\beta^2}{n-(n-2)\beta} + \beta\right) \end{aligned}$$

$$\times\left(w^*+\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\Bigg]$$
$$=\frac{1-p_0}{n}\left[\beta w^*+(n-\beta)\left(\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\right]$$
$$+\frac{p_0}{n}\Bigg[nu_i(1)+\frac{n\beta}{n-(n-2)\beta}w^*+\frac{n\beta}{n-(n-2)\beta}$$
$$\times\left(\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}\right)\Bigg]. \quad (13)$$

Of course, we can get rid of p_0, whose value we do not know, writing instead of (13) the following two inequalities, at least one of which is implied by (13) (as we shall see each of them will be enough to obtain the desired conclusion):

$$\begin{cases} w^*\le\frac{1}{n}[\beta w^*+(n-\beta)(\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)})],\\ w^*\le\frac{1}{n}[nu_i(1)+\frac{n\beta}{n-(n-2)\beta}w^*+\frac{n\beta}{n-(n-2)\beta}(\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)})]\end{cases}$$

or equivalently

$$\begin{cases} w^*\le\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)},\\ w^*\le\frac{n-(n-2)\beta}{n-(n-1)\beta}u_i(1)+\frac{\beta}{n-(n-1)\beta}(\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}).\end{cases} \quad (14)$$

Fix now $\beta\in(\frac{1}{2},1)$ and suppose that playing bold is optimal for i in x^*, that is $w^*\ge v^*$, which together with (10) gives

$$w^*\ge\frac{n+\beta(n-1)}{(2-\beta)n}u_i(1).$$

This by (14) implies that one of the following two inequalities holds:

$$\begin{cases} \frac{n+\beta(n-1)}{(2-\beta)n}u_i(1)\le\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)},\\ \frac{n+\beta(n-1)}{(2-\beta)n}u_i(1)\le\frac{n-(n-2)\beta}{n-(n-1)\beta}u_i(1)+\frac{\beta}{n-(n-1)\beta}(\frac{nu_i(1)}{n-2\beta}+\frac{2(n-2)\beta^2u_i(1)}{(n-2\beta)(n-(n-2)\beta)(1-\beta)}).\end{cases}$$

Now note that the limit as $n\to\infty$ of the LHS of the above inequalities is $\frac{1+\beta}{2-\beta}u_i(1)$, while the limits of each of the RHSs are equal to $u_i(1)$. If we take an n such that all the above terms are within $\frac{2\beta-1}{2(2-\beta)}u_i(1)>0$ from their limits, we obtain in both inequalities that:

$$\frac{1+\beta}{2-\beta}u_i(1)+\frac{2\beta-1}{2(2-\beta)}u_i(1)\le u_i(1)-\frac{2\beta-1}{2(2+\beta)}u_i(1).$$

This can be rewritten as

$$2(1+\beta)+(2\beta-1)\le 2(2-\beta)-(2\beta-1).$$

But this is only possible if $\beta \le \frac{1}{2}$, contradicting the assumption that $\beta > \frac{1}{2}$. This means that playing bold cannot be optimal for i in state x^*. □

5 Proof of Theorem 2

The proof is an immediate consequence of Theorem 1 and Lemma 3.

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